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INJECTIVE SPACES OF REAL-VALUED FUNCTIONS WITH THE BAIRE PROPERTY

BY

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ABSTRACT

Generalizing the technique used by S.A. Argyros in [3], we give a lemma from which certain Banach spaces are shown to be non-injective. This is applied mainly to study the injectivity of spaces of real-valued Borel functions and functions with the Baire property on a topological space. The results obtained in this way do not follow from previous works about this matter.

Introduction

A real Banach space will be called **injective** if for every Banach space Z containing X as a subspace, there exists a projection from Z onto X. The class of injective Banach spaces of the form $C(X)$ has been investigated by Amir [1] and [2], Isbell and Semadeni [6] and Wolfe in [10] and [11].

Argyros, solving a question of Rosenthal, proved in [3] that the space B of all bounded Borel functions on $[0, 1]$ and the space L of all bounded Lebesgue measurable functions on $[0, 1]$ are not injective spaces. Although these spaces L and B admit an isometric representation as a $C(K)$, where K is a zero-dimensional compactification of a discrete space of cardinality $2^{\aleph_0}$, the Amir boundary of K [10] is extremely disconnected, therefore Argyros' result does not follow from the cited papers.

In this note we generalize the technique of Argyros and prove that, in very general situations, the Banach space $B(X)$ of all bounded real-valued functions

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with the Baire property on a topological space X is not injective. Let us notice that the spaces $B(X)$ considered here are isometric to a space $C(K)$, being K, a compact space whose Amir boundary is extremally disconnected.

The results

For a set X, $c_0(X)$ will be the closure in $l_\infty(X)$ of the linear span of the set of all characteristic functions of the one-point subsets of X. Thus, $c_0(X)$ is a Banach space endowed with the supremum-norm.

As usual, $|X|$ will be the cardinal number of X, i.e. the least ordinal number which can be put in one-one correspondence with the set X.

By an ideal $\mathcal{I}$ on a set X we mean an ideal on the power set of X. We say that $\mathcal{G}$ is a generator of $\mathcal{I}$ if $\mathcal{G}$ is a subset of $\mathcal{I}$ such that every set in $\mathcal{I}$ is contained in some set of $\mathcal{G}$. The members of $\mathcal{G}$ are also called generators of $\mathcal{I}$.

The proof of the following lemma is essentially the same that Argyros' proof of lemma 3 in [3]. However, in order to show the variations appearing in general case, we include the complete proof.

LEMMA 1 (Argyros): *Let $\mathcal{I}$ be a σ-ideal on a set X which is contained in a σ-algebra Σ of subsets of X and suppose that the following conditions are satisfied:*

(a) *$\mathcal{I}$ contains every one-point subset of X.*

(b) *Every point-finite family of members of $\Sigma \setminus \mathcal{I}$ is countable*

(c) *There exists a generator $\mathcal{G}$ of $\mathcal{I}$ such that $|X \setminus G| \geq |\mathcal{G}| > \aleph_0$, for all $G \in \mathcal{G}$.*

If $T: l_\infty(X) \to l_\infty(X)$ is a bounded linear operator whose restriction to $c_0(X)$ is the identity, then there is a function $f \in l_\infty(X)$ such that $T(f)$ is not Σ-measurable.

Proof: Let $\mathfrak{m} = |\mathcal{G}|$. We say that a subset B of X is **thick** if $|B \setminus G| \geq \mathfrak{m}$, for all $G \in \mathcal{G}$.

CLAIM:

(1) If B is a thick subset of X and $J \in \mathcal{I}$, then $B \setminus J$ is thick.

(2) Every thick subset of X can be decomposed into $\mathfrak{m}$ pairwise disjoint thick subsets.

Proof: (1) is trivial. In order to prove (2) take an enumeration $\{G_\alpha\}_{\alpha<\mathfrak{m}}$ of $\mathcal{G}$ such that each generator is listed cofinally $\mathfrak{m}$ times.

Given a thick set B, put $E_\alpha = B \setminus G_\alpha$, for each $\alpha < \mathfrak{m}$. We proceed now by transfinite induction. At the step α choose distinct points $x_{\alpha\beta}$, for each $\beta \leq \alpha$, from $E_\alpha \setminus \{x_{\gamma\delta} |\ \delta \leq \gamma < \alpha\}$. This is always possible because $|E_\alpha| \geq \mathfrak{m}$.

Define $Y_\beta = \{x_{\alpha\beta} |\ \beta < \alpha < \mathfrak{m}\}$. These sets are clearly pairwise disjoint and fixing a generator $G \in \mathcal{G}$, for each α such that $G = G_\alpha$, we have $x_{\alpha\beta} \in Y_\beta \cap E_\alpha$ and therefore, since G appears $\mathfrak{m}$ times, it follows that $|Y_\beta \cap X \setminus G| \geq \mathfrak{m}$. Thus each Y_β is a thick subset of B.

To get a partition we simply add the possible remainder $B \setminus \bigcup_{\alpha<\mathfrak{m}} Y_\alpha$ to Y_0 and so we guarantee that $B = \bigcup_{\alpha<\mathfrak{m}} Y_\alpha$. The claim is proved.

If T^* is the conjugate of the operator T and for each $x \in X$, δ_x is the Dirac measure with support $\{x\}$, then $\mu_x = T^*(\delta_x)$ is a finitely additive measure on X. Moreover, $\| \mu_x \| = |\mu_x|(X)$ and $\| T \| = \sup_{x\in X} \| \mu_x \|$.

As X is trivially thick, there is a partition of X into $\mathfrak{m}$ pairwise disjoint thick subsets $\{B^1_\alpha\}_{\alpha<\mathfrak{m}}$. For each $\alpha < \mathfrak{m}$ define

$$H^1_\alpha = \Big\{x \in B^1_\alpha |\ |\mu_x|\Big(\bigcup_{\beta\neq\alpha} B^1_\beta\Big) < 1/2\Big\}.$$

There are two possibilities:

CASE 1: For all $\alpha < \mathfrak{m}$ the set H^1_α is not in $\mathcal{I}$.

CASE 2: There exists an $\alpha_1 < \mathfrak{m}$ such that $H^1_{\alpha_1}$ is in $\mathcal{I}$.

If case 2 holds, since $B^1_{\alpha_1} \setminus H^1_{\alpha_1}$ is thick, it can be decomposed into $\mathfrak{m}$ pairwise disjoint thick subsets $\{B^2_\alpha\}_{\alpha<\mathfrak{m}}$ and we can define again

$$H^2_\alpha = \Big\{x \in B^2_\alpha |\ |\mu_x|\Big(\bigcup_{\beta\neq\alpha} B^2_\beta\Big) < 1/2\Big\}.$$

We repeat this process as long as case 2 holds, but case 1 must occur after a finite number of steps. To see this, consider a natural number $n > 2 \| T \|$ and suppose that case 2 has held up to step $n-1$. Thus we have defined the sets H^n_α and now case 2 cannot hold, for if some $H^n_{\alpha_n} \in \mathcal{I}$, then $B^n_{\alpha_n} \setminus H^n_{\alpha_n}$ is thick and, in particular, it is non-empty. Take $x \in B^n_{\alpha_n} \setminus H^n_{\alpha_n}$. Since $x \in B^j_{\alpha_j} \setminus H^j_{\alpha_j}$ for all $j = 1, \ldots, n$, there are n pairwise disjoint sets with measure $|\mu_x|$ greater than or equal to $1/2$, and because of finite additivity of $|\mu_x|$, we conclude that $|\mu_x|(X) \geq n/2 > \| T \|$, but this is a contradiction.

Consequently, if case 1 holds at the step m, the sets

$$H^m_\alpha = \Big\{x \in B^m_\alpha |\ |\mu_x|\Big(\bigcup_{\beta\neq\alpha} B^m_\beta\Big) < 1/2\Big\} \quad \text{for } \alpha < \mathfrak{m}$$

are pairwise disjoint and none of them belongs to $\mathcal{I}$.

Enumerate $\mathcal{G} = \{\mathcal{G}_\alpha\}_{\alpha<\mathfrak{m}}$. Clearly $H_\alpha^m \setminus G_\alpha \notin \mathcal{I}$, and so it contains an uncountable subset $\{x_\eta^\alpha\}_{\eta<\aleph_1}$. Define $F_\eta = \{x_\eta^\alpha | \alpha < \mathfrak{m}\}$. Thus we have:

1. $F_\eta \setminus G_\alpha \neq \emptyset$ for all α, so that $F_\eta \notin \mathcal{I}$.

2. If $x \in F_\eta$, then $|\mu_x|(F_\eta \setminus \{x\}) < 1/2$, for there is an α such that $x \in H_\alpha^m$, which implies that $F_\eta \setminus \{x\} \subset \bigcup_{\beta\neq\alpha} B_\beta^m$ and we know $|\mu_x|(\bigcup_{\beta\neq\alpha} B_\beta^m) < 1/2$.

3. F_η is a subset of $V_\eta = \{x \in X | \, |T(\chi_{F_\eta})(x)| > 1/2\}$, for if $x \in F_\eta$, by the hypothesis on T, $\mu_x(\{x\}) = 1$ and then

$$|T(\chi_{F_\eta})(x)| = |\mu_x(F_\eta)| \geq |\mu_x(\{x\})| - |\mu_x(F_\eta \setminus \{x\})| > 1/2.$$

As the sets $F\eta$ are pairwise disjoint and T is bounded, the family $\{V_\eta\}_{\eta<\aleph_1}$ is point-finite. From facts 1 and 3 we get that $V_\eta \notin \mathcal{I}$, for all $\eta < \aleph_1$. If every $T(\chi_{F_\eta})$ were Σ–measurable, we would have $V_\eta \in \Sigma \setminus \mathcal{I}$ for all η, which would contradict part (c) of the hypotheses. So at least one $T(\chi_{F_\eta})$ must be non-measurable and this concludes the proof.

Remark: Notice that Argyros' result follows from our Lemma 1 taking X the unit interval, Σ the σ-algebra of Lebesgue-measurable sets and $\mathcal{I}$ the σ-ideal of null sets. In fact Lemma 1 can be applied to any uncountable completely metrizable separable space with a non trivial regular σ-finite measure in which points are null sets.

As an immediate application of Lemma 1, we present the following example. Recall that an infinite cardinal $\mathfrak{m}$ is **regular** if every unbounded subset of $\mathfrak{m}$ has cardinality $\mathfrak{m}$.

Example 2: Let X be a set of cardinality $\mathfrak{m}$ and let $\mathfrak{p}$ be another cardinal $\mathfrak{p} \leq \mathfrak{m}$, let $\mathcal{I}_\mathfrak{p}$ be the σ-ideal consisting of all subsets of X with cardinality less than $\mathfrak{p}$, let $\mathcal{F}_\mathfrak{p} = \{X \setminus A | \, A \in \mathcal{I}_\mathfrak{p}\}$ be its dual filter and, finally, let $\Sigma_\mathfrak{p} = \mathcal{I}_\mathfrak{p} \cup \mathcal{F}_\mathfrak{p}$ be the σ-algebra generated by $\mathcal{I}_\mathfrak{p}$.

If $\mathfrak{m}$ is an uncountable regular cardinal, then $\Sigma_\mathfrak{m}$, $\mathcal{I}_\mathfrak{m}$, satisfy parts (a), (b) and (c) of Lemma 1. Condition (b) holds because there are no infinite point-finite families of members of $\Sigma \setminus \mathcal{I}$. In order to check (c) we can suppose that $X = [1, \mathfrak{m})$ and, in this case, the sets $[1, \alpha)$ with $\alpha < \mathfrak{m}$ form a system of $\mathfrak{m}$ generators for $\mathcal{I}_\mathfrak{m}$.

We thus obtain that there exists no projection from $l_\infty(X)$ to the Banach space $M(\Sigma_\mathfrak{m})$ of all $\Sigma_\mathfrak{m}$-measurable bounded functions and therefore $M(\Sigma_\mathfrak{m})$ is not an injective space.

It is not difficult to prove that $M(\Sigma_{\mathfrak{p}})$ consists of all real-valued functions which are constant on a set A belonging to $\mathcal{F}_{\mathfrak{p}}$. In particular if $X = [1, \omega_1]$, then $M(\Sigma_{\aleph_1})$ is the space $\mathrm{Ba}(X)$ of all real-valued Baire functions on X considered as a topological space with the order topology. So this space is not injective.

Remark: It should be noticed that, in fact, none of the spaces $M(\Sigma_{\mathfrak{p}})$ for $\mathfrak{p} \leq \mathfrak{m}$ is injective. This is because in this particular case, the proof of Lemma 1 goes on without the assumption (c). We simply define a thick set as a set B with $|B| = |X \setminus B| = \mathfrak{m}$ and skip every mention of generators. The sets F_η have cardinality $\mathfrak{m}$ and so they do not belong to $\mathcal{I}$.

Moreover, if $X = \mathbb{N}$, the set of positive integers, the argument remains valid up to obtaining the infinite sets V_η and then we see that the sequences $T(\chi_{F_\eta})$ cannot converge to zero, because then V_η would be finite. This gives a proof of the well-known result of Philips [9], which asserts that there is no projection from l_∞ to c_0. In fact, working with arbitrary cardinals $\mathfrak{p} \leq \mathfrak{m}$ we obtain a more elementary proof of the generalizations of Philips Theorem given by A. Pełczyńsky and V.N. Sudakov in ([8], Th. 1, Cor. 2).

By a **space**, we mean a topological Hausdorff space. The **weight** $w(X)$ of a space X is the minimum cardinal of a basis for the open sets of X. A subset of a space X has the **Baire property** if there is an open set G in X such that $(A \setminus G) \cup (G \setminus A)$ is of first category. The set Σ of all subsets of X with the Baire property is a σ-algebra and it contains the σ-ideal of all sets of first category. We write $B(X)$ for the set of all real-valued bounded functions with the Baire property, i.e. Σ-measurable.

PROPOSITION 3: *Let X be a Baire space with the countable chain condition, without isolated points, such that $w(X)^{\aleph_0} \leq |X|$ and in which every dense G_δ subset has cardinality $|X|$. Then there is no continuous operator from $l_\infty(X)$ to $B(X)$ whose restriction to $c_0(X)$ is the identity.*

Proof: Take Σ to be the σ-algebra of all subsets of X with the Baire property and $\mathcal{I}$ the σ-ideal of all sets of first category. Let $\mathcal{B}$ a basis for the open sets of X such that $|\mathcal{B}| = w(X)$.

If E is a closed set with empty interior, there is a family $\{B_i\}_{i \in J}$ of pairwise disjoint members of $\mathcal{B}$ contained in $X \setminus E$ and maximal with respect to this property. The set J must be countable because X satisfies the countable chain condition.

By maximality $\bigcup_{i\in J} B_i$ is dense in X and therefore $X \setminus \bigcup_{i\in J} B_i$ has empty interior. Put

$$\mathcal{G} = \{\bigcup_{n=0}^{\infty} F_n\colon \text{ each } F_n \in \mathcal{I} \text{ and } X \setminus F_n$$
$$\text{is a dense countable union of members of } \mathcal{B}\}.$$

We have proved that $\mathcal{G}$ is a generator of $\mathcal{I}$ and by hypothesis $|\mathcal{G}| \le w(X)^{\aleph_0} \le |X|$.

On the other hand, if A is a member of $\mathcal{G}$, then $X \setminus A$ is a countable intersection of dense open sets. Thus, since X is a Baire space, $X \setminus A$ is a dense G_δ subset. By hypothesis again, $|X \setminus A| = |X|$.

So, condition (c) of Lemma 1 is fulfilled, condition (a) is evident and condition (b) holds by ([5], 1Eb, 7Bd). Therefore the result follows.

COROLLARY 4: *If X is the product space $Y^{\mathfrak{m}}$, where Y is a separable compact space with more than one point and $\mathfrak{m}$ is uncountable, then $B(X)$ is not injective.*

Proof: By ([4], Cor 2.3.18), X has the countable chain condition. Since X is compact, $w(X) \le |X|$ ([4], Th. 3.1.21), and since $|X|^{\aleph_0} = |X|$, we have $w(X)^{\aleph_0} \le |X|$. Finally, the non-empty G_δ subsets of X are easily shown to have cardinality $|X|$ and therefore corollary follows from Proposition 3.

Recall that a point of a space is a **condensation point** if all its neighborhoods are uncountable.

THEOREM 5: *Let X be a completely metrizable separable space. The following statements are equivalent:*

(1) *There is an injective Banach space between $c_0(X)$ and $B(X)$.*
(2) *The set C of condensation points of X has empty interior.*
(3) $B(X) = l_\infty(X)$.

Proof: (1) $\to$ (2) Suppose that $U = \operatorname{int} C$ is non-empty. Then U is a separable completely metrizable space without isolated points. If F is a dense G_δ subset of U then F is an uncountable Borel set and hence it contains a copy of the Cantor set ([7], p. 447). Therefore $|F| = 2^{\aleph_0} = |X|$ and, according to Proposition 3, there is no bounded operator from $l_\infty(U)$ to $B(U)$ whose restriction to $c_0(U)$ is the identity.

Now suppose that E is an injective Banach subspace of $B(X)$ containing $c_0(X)$. Then there is a projection $P\colon l_\infty(X) \to E$. For each $h \in l_\infty(U)$ let $T(h)$ be the extension of h to X which vanishes on $X \setminus U$. Clearly $T\colon l_\infty(U) \to l_\infty(X)$ is a continuous extension operator such that $T(B(U))$ is contained in $B(X)$. Let $\overline{P} = rPT$, where r is the restriction to U. Thus $\overline{P}\colon l_\infty(U) \to B(U)$ is a continuous operator whose restriction to $c_0(U)$ is the identity. This is a contradiction.

(2) $\to$ (3) Let A be the set of isolated points of X and let us show that A is dense in X. If U is a non-empty open set disjoint from A, since C has empty interior, the set $V = U \setminus C$ is open and non-empty. Then, V being a separable completely metrizable space without isolated points, we conclude that it is uncountable, but this is impossible since $X \setminus C$ is countable. Consequently A is dense in X and therefore $X \setminus A$ is of first category. Since A is discrete, by ([7], p. 400) $B(X) = l_\infty(X)$.

(3) $\to$ (1) is trivial.

Corollary 6: *Let X be a separable completely metrizable space. The following statements are equivalent:*

(1) *The space* Bo(X) *of all real-valued bounded Borel measurable functions on X is injective.*

(2) *X is countable.*

(3) Bo$(X) = l_\infty(X)$.

Proof: (1)$\to$(2) If X is uncountable, then X contains a copy $\mathcal{C}$ of the Cantor set ([7], p. 447). For each $h \in l_\infty(\mathcal{C})$, let $T(h)$ be the extension of h to X which vanishes on $X \setminus \mathcal{C}$. Then T is a continuous extension operator from $l_\infty(\mathcal{C})$ to $l_\infty(X)$ such that $T(\mathrm{Bo}(\mathcal{C}))$ is contained in Bo(X).

Suppose that Bo(X) is injective and let P a projection from $l_\infty(X)$ onto Bo(X). Let r be the restriction to $\mathcal{C}$. Thus, since $r(\mathrm{Bo}(X)) = \mathrm{Bo}(\mathcal{C})$, $\overline{P} = rPT$ is a projection from $l_\infty(\mathcal{C})$ onto Bo$(\mathcal{C})$ and therefore Bo$(\mathcal{C})$ is injective, but this is impossible by Theorem 5.

Implications (2) $\to$ (3) and (3) $\to$ (1) are both trivial.

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